An Introduction to Quantum Groups and their Representation Theory: $U_q(\mathfrak{sl}_2)$

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Defining $U_q(\mathfrak{sl}_2)$

Let $\mathbb{F}$ be a field, and consider $\mathbb{F}(q)$ where $q^2 \neq 1$. Then

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\begin{aligned}
KK^{-1} &= 1 = K^{-1}K \\
KEK^{-1} &= q^2 E \\
KFK^{-1} &= q^{-2} F \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}
\end{aligned}
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Note: $q^2 \neq 1$ is required in the last relation, but also implies from the middle two that $U_q(\mathfrak{sl}_2)$ is a noncommutative algebra.

Representations of $\mathfrak{g}$ and G

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$$(x \cdot f)(u) := f(-x \cdot u), \quad (g \cdot f)(u) := f(g^{-1} \cdot u)$$

Representations of $\mathfrak{g}$ and G (cont.)

Note that these actions allow one to define a $\mathfrak{g}$-module (resp. G-module) structure on $\mathrm{Hom}(U, V)$ by utilizing the isomorphism $\mathrm{Hom}(U, V) \cong U^* \otimes V$ i.e.

$$(x \cdot f)(u) := x \cdot (f(u)) - f(x \cdot u), \quad (g \cdot f)(u) := g \cdot (f(g^{-1} \cdot u))$$

Representations of $\mathfrak{g}$ and G (cont.)

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Note: With this definition of $\mathfrak{g}$-module (resp. G-module) structure on $\mathrm{Hom}(U, V)$, the aforementioned isomorphism of vector spaces becomes an isomorphism of $\mathfrak{g}$-modules (resp. G-modules).

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- I will say module and representation interchangeably, but what matters is the map!
- My modules are finite-dimensional.
- All of my tensor products are over $\mathbb{C}$.
- A $U_q(\mathfrak{sl}_2)$-module (M, ϕ) is an algebra homomorphism $\phi : U_q(\mathfrak{sl}_2) \to \mathrm{End}(M)$

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$$\phi : U_q(\mathfrak{sl}_2) \to \mathrm{End}(M), \quad \psi : U_q(\mathfrak{sl}_2) \to \mathrm{End}(N)$$

are representations, we at least have the algebra homomorphism

$$\phi \otimes \psi : U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \to \mathrm{End}(M) \otimes \mathrm{End}(N), \quad x \otimes y \mapsto \phi(x) \otimes \psi(y)$$

Which then by the above identification gives us a map on $M \otimes N$.

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Which then by the above identification gives us a map on $M \otimes N$. Thus, if we can construct an algebra homomorphism from $U_q(\mathfrak{sl}_2)$ to its tensor square, we can take the composition and make the tensor product of modules into a module.

Representations of $U_q(\mathfrak{sl}_2)$: A Comultiplication (cont.)

Representations of $U_q(\mathfrak{sl}_2)$: A Comultiplication (cont.)

Lemma

There is a unique algebra homomorphism $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ *given by*

$$\Delta(E) := E \otimes 1 + K \otimes E$$
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Note: The first two formulas are eerily similar to how Lie algebras act on tensor products, while the last is how groups act on tensor products.

(Sketch of) Proof.

As this map is defined on the generators of $U_q(\mathfrak{sl}_2)$, we need only check that the images $\Delta(E), \Delta(F), \Delta(K^{\pm 1})$ preserve the defining relations of $U_q(\mathfrak{sl}_2)$. I will show it for the last relation, leaving it to the interested viewer to check the rest!

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Recall that the fourth relation states

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Thus, we need to prove that taking Δ of both sides gives the same result.

(Sketch of) Proof (cont.)

On the left hand side we have

$$\Delta(EF - FE) = \Delta(E)\Delta(F) - \Delta(F)\Delta(E)$$

(Sketch of) Proof (cont.)

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$$\begin{aligned}\Delta(EF - FE) &= \Delta(E)\Delta(F) - \Delta(F)\Delta(E) \\ &= (E \otimes 1 + K \otimes E)(F \otimes K^{-1} + 1 \otimes F) \\ &\quad - (F \otimes K^{-1} + 1 \otimes F)(E \otimes 1 + K \otimes E)\end{aligned}$$

(Sketch of) Proof (cont.)

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(Sketch of) Proof (cont.)

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&\quad - (F \otimes K^{-1} + 1 \otimes F)(E \otimes 1 + K \otimes E) \\
&= EF \otimes K^{-1} + E \otimes F + KF \otimes EK^{-1} + K \otimes EF \\
&\quad - FE \otimes K^{-1} - FK \otimes K^{-1}E - E \otimes F - K \otimes FE \\
&= (EF - FE) \otimes K^{-1} + K \otimes (EF - FE) \\
&\quad + KF \otimes EK^{-1} - FK \otimes K^{-1}E
\end{aligned}$$

(Sketch of) Proof (cont.)

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&= EF \otimes K^{-1} + E \otimes F + KF \otimes EK^{-1} + K \otimes EF \\
&\quad - FE \otimes K^{-1} - FK \otimes K^{-1}E - E \otimes F - K \otimes FE \\
&= (EF - FE) \otimes K^{-1} + K \otimes (EF - FE) \\
&\quad + KF \otimes EK^{-1} - FK \otimes K^{-1}E \\
&= (EF - FE) \otimes K^{-1} + K \otimes (EF - FE) \\
&\quad + q^2 q^{-2} FK \otimes K^{-1}E - FK \otimes K^{-1}E
\end{aligned}$$

(Sketch of) Proof (cont.)

Thus,

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and indeed the relation holds, as well as the others. □

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and indeed the relation holds, as well as the others. □

We call $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ a comultiplication. More on why in a few slides!

Representations of $U_q(\mathfrak{sl}_2)$: A Counit

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However, $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$ as algebras i.e. an endomorphism of $\mathbb{C}$ is the same thing as picking a scalar.

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Lemma

There is a unique algebra homomorphism $\varepsilon : U_q(\mathfrak{sl}_2) \to \mathbb{C}$ given by

$$\varepsilon(E) := 0$$
$$\varepsilon(F) := 0$$
$$\varepsilon(K) := 1$$

We call ε a counit. More on why in a few slides!

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We call ε *a counit. More on why in a few slides!*

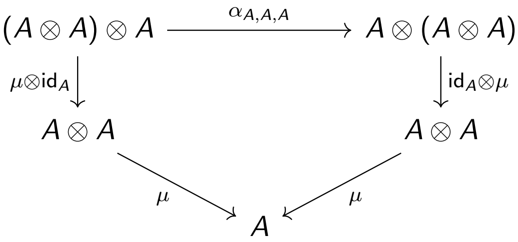
Note: Yet again, the first two formulas are eerily similar to how Lie algebras act on the ground field, while the last is how groups act on the ground field.

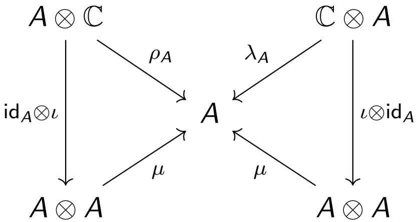
On Comultiplications and Counits

On Comultiplications and Counits

One of the equivalent ways of defining an associative, unital $\mathbb{C}$-algebra is as a triple (A, μ, ι) consisting of a $\mathbb{C}$-vector space A and $\mathbb{C}$-linear maps $\mu : A \otimes A \to A$ and $\iota : \mathbb{C} \to A$ such that the following diagrams commute:

On Comultiplications and Counits (cont.)





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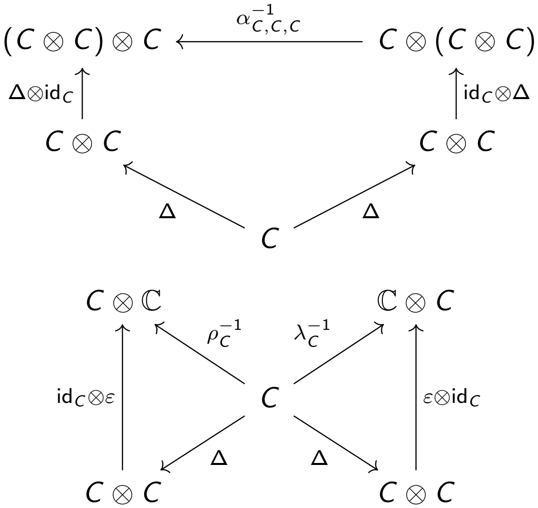
On Comultiplications and Counits (cont.)

It might be new for some, but perhaps not surprising that there is a related object one can define called a coassociative, counital $\mathbb{C}$-coalgebra. Would anyone like to guess its definition?

It is a triple (C, Δ, ε) consisting of a $\mathbb{C}$-vector space C and $\mathbb{C}$-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{C}$ such that the following diagrams commute:

On Comultiplications and Counits (cont.)



On Comultiplications and Counits (cont.)

It is then a computational check on the generators to show that $(U_q(\mathfrak{sl}_2), \Delta, \varepsilon)$ is a coassociative, counital $\mathbb{C}$-coalgebra! But the previous lemmas said something more about the comultiplication and counit.

On Comultiplications and Counits (cont.)

It is then a computational check on the generators to show that $(U_q(\mathfrak{sl}_2), \Delta, \varepsilon)$ is a coassociative, counital $\mathbb{C}$-coalgebra! But the previous lemmas said something more about the comultiplication and counit. These maps also happen to be algebra homomorphisms! This leads to the following definition

Bialgebras

Definition

A bialgebra is a quintuple $(B, \mu, \iota, \Delta, \varepsilon)$ such that (B, μ, ι) is an associative, unital algebra, (B, Δ, ε) is a coassociative, counital coalgebra, and such that any two of the equivalent conditions hold

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1. Δ and ε are algebra homomorphisms.
2. μ and ι are coalgebra homomorphisms.

Representations of $U_q(\mathfrak{sl}_2)$: An Antipode

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Let us recall how Lie algebras and groups act on duals:

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Given a module M we want to turn $M^* := \mathrm{Hom}(M, \mathbb{C})$ into a $U_q(\mathfrak{sl}_2)$-module as well.
Let us recall how Lie algebras and groups act on duals:

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Lemma

There is a unique algebra antiautomorphism $S : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ given by

$$S(E) := -K^{-1}E$$
$$S(F) := -FK$$
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We call S an antipode.

Representations of $U_q(\mathfrak{sl}_2)$: An Antipode (cont.)

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Note: Yet again, the first two formulas are eerily similar to how Lie algebras act on duals, while the last is how groups act on duals.

Representations of $U_q(\mathfrak{sl}_2)$: An Antipode (cont.)

Thus, if (M, ϕ) is a $U_q(\mathfrak{sl}_2)$-module, then M^* can be given a $U_q(\mathfrak{sl}_2)$-module structure by defining

$$\phi^S : U_q(\mathfrak{sl}_2) \to \mathrm{End}(M^*), \quad x \mapsto (f \mapsto f \circ \phi(S(x)))$$

On the Antipode

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Lemma

The following diagrams commute

$$\begin{array}{ccc} U_q(\mathfrak{sl}_2) & \xrightarrow{\Delta} & U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \\ \downarrow{\scriptstyle \iota\circ\varepsilon} & & \downarrow{\scriptstyle id\otimes S} \\ U_q(\mathfrak{sl}_2) & \xleftarrow[\mu]{} & U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \end{array} \qquad \begin{array}{ccc} U_q(\mathfrak{sl}_2) & \xrightarrow{\Delta} & U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \\ \downarrow{\scriptstyle \iota\circ\varepsilon} & & \downarrow{\scriptstyle S\otimes id} \\ U_q(\mathfrak{sl}_2) & \xleftarrow[\mu]{} & U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \end{array}$$

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(Sketch of) Proof.

One proceeds to verify this on the generators of $U_q(\mathfrak{sl}_2)$ and that $\iota \circ \varepsilon$ is multiplicative; then the commutativity indeed follows. Proving the equality on the generators is tedious computation, but the latter part is delicate. The reason it is not readily true is ι is not an algebra homomorphism, nor are S and μ. □

Hopf Algebras

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Definition

A Hopf algebra is a sextuple $(H, \mu, \iota, \Delta, \varepsilon, S)$ such that $(H, \mu, \iota, \Delta, \varepsilon)$ is a bialgebra, and such that the following diagrams commute

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \iota\circ\varepsilon}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes S} \\ H & \xleftarrow[\mu]{} & H \otimes H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \iota\circ\varepsilon}\downarrow & & \downarrow{\scriptstyle S\otimes \mathrm{id}} \\ H & \xleftarrow[\mu]{} & H \otimes H \end{array}$$

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$(U_q(\mathfrak{sl}_2), \mu, \iota, \Delta, \varepsilon, S)$ as defined throughout this talk is a Hopf algebra!

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$(U_q(\mathfrak{sl}_2), \mu, \iota, \Delta, \varepsilon, S)$ as defined throughout this talk is a Hopf algebra!
Note: The antipode is uniquely determined by having to satisfy the commutativity of the above two diagrams. The map will automatically be an antihomomorphism, but might not necessarily be invertible.

What About Lie Algebras and Groups?

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Modules of a Lie algebra $\mathfrak{g}$ (resp. a group G) are equivalent to modules of its universal enveloping algebra $U(\mathfrak{g})$ (resp. group algebra $\mathbb{C}[G]$). In particular, both of these associative, unital algebras admit the following respective Hopf algebra structure

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- $U(\mathfrak{g})$
  - $\Delta : x \mapsto x \otimes 1 + 1 \otimes x$

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Note: For group algebras, the group elements are typically denoted alternatively to not confuse the operations of the algebra with that of the group e.g. writing δ_g to represent the basis element corresponding to $g \in G$.

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Suppose (U, ϕ) and (V, ψ) are $\mathfrak{g}$-modules, hence $U(\mathfrak{g})$-modules. Then I claim their tensor product $U \otimes V$ becomes a $U(\mathfrak{g})$-module by utilizing the comultiplication. And indeed, following the schematic from quantum groups:

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$$x \xmapsto{\Delta} x \otimes 1 + 1 \otimes x \xmapsto{\phi \otimes \psi} \phi(x) \otimes \mathrm{id}_V + \mathrm{id}_U \otimes \psi(x)$$

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Which is exactly what we saw in the beginning!

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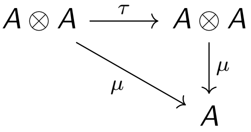

where $\tau : x \otimes y \mapsto y \otimes x$.

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An equivalent way to define a commutative algebra is to require the following diagram to commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow^{\mu} & \downarrow \mu \\ & & A \end{array}$$

where $\tau : x \otimes y \mapsto y \otimes x$. Unsurprisingly, a coalgebra is then called cocommutative if the dual diagram commutes:

$$\begin{array}{ccc} C \otimes C & \xleftarrow{\tau} & C \otimes C \\ & \nwarrow^{\Delta} & \uparrow \Delta \\ & & C \end{array}$$

Cartier-Gabriel-Kostant theorem

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Theorem

Any cocommutative Hopf algebra H over an algebraically closed field $\mathbb{K}$ of characteristic zero is of the form

$$\mathbb{K}[G] \ltimes U(\mathfrak{g}),$$

where $\mathfrak{g}$ is a Lie algebra and G is a group acting on $\mathfrak{g}$.

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where $\mathfrak{g}$ is a Lie algebra and G is a group acting on $\mathfrak{g}$.

Note: $\mathfrak{g}$ is actually the Lie algebra of primitive elements of H, and G is the group of group-like elements of H.

What about Hom sets?

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As with the dual space, let us turn to the Lie algebra and group rep. theory picture for motivation. To that end, recall that $\mathrm{Hom}(U, V)$ becomes a $\mathfrak{g}$-module (resp. G-module) with the following action induced from that on U and V

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Now that we are aware of comultiplications and antipodes, we can manipulate the above action to see what the picture should be for $U_q(\mathfrak{sl}_2)$. Let us do so for the Lie algebra case.

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$$\begin{aligned}(x \cdot f)(u) &:= x \cdot (f(u)) - f(x \cdot u) \\ &= x \cdot (f(1 \cdot u)) + 1 \cdot f(S(x) \cdot u)\end{aligned}$$

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where really the second isomorphism follows from the first since $\tau : V \otimes U^* \to U^* \otimes V$ is an isomorphism of vector spaces.
Note: The $\mathfrak{g}$-modules (resp. G-module) structure on $\mathrm{Hom}(U, V)$ makes these isomorphisms become isomorphisms of $\mathfrak{g}$-modules (resp. G-modules).

The story of Hom(U, V) for $U_q(\mathfrak{sl}_2)$

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makes the previous map $V \otimes U^* \to \mathrm{Hom}(U, V)$ an isomorphism of $U_q(\mathfrak{sl}_2)$-modules. If instead we take

$$(x \cdot f)(u) := \sum_i x_i' \cdot f(S(x_i) \cdot u)$$

then $U^* \otimes V \to \mathrm{Hom}(U, V)$ becomes an isomorphism of $U_q(\mathfrak{sl}_2)$-modules.

Symmetry of Tensor Products

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The transposition map is no longer an isomorphism for $U_q(\mathfrak{sl}_2)$-modules! The natural approach to just swapping tensor factors will not in general preserve the $U_q(\mathfrak{sl}_2)$ structure. We thus need something else to be able to swap tensor factors in place of τ. To that end, what really makes τ so special?

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 & \overset{\mathrm{id}_1 \otimes \tau_{2,3}}{\nearrow} & & & & \overset{\tau_{1,3} \otimes \mathrm{id}_2}{\searrow} & \\
V_1 \otimes (V_2 \otimes V_3) & & & & & & (V_3 \otimes V_1) \otimes V_2 \\
 & \underset{\alpha^{-1}_{1,2,3}}{\searrow} & & & & \underset{\alpha^{-1}_{3,1,2}}{\nearrow} & \\
 & & (V_1 \otimes V_2) \otimes V_3 & \xrightarrow[\tau_{12,3}]{} & V_3 \otimes (V_1 \otimes V_2) & &
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V_1 \otimes (V_2 \otimes V_3) & & & & & & (V_3 \otimes V_1) \otimes V_2 \\
 & \searrow_{\alpha^{-1}_{1,2,3}} & & & & \nearrow_{\alpha^{-1}_{3,1,2}} & \\
 & & (V_1 \otimes V_2) \otimes V_3 & \xrightarrow[\tau_{12,3}]{} & V_3 \otimes (V_1 \otimes V_2) & &
\end{array}$$

and

$$\begin{array}{ccccccc}
 & & (V_2 \otimes V_1) \otimes V_3 & \xrightarrow{\alpha_{2,1,3}} & V_2 \otimes (V_1 \otimes V_3) & & \\
 & \nearrow^{\tau_{1,2} \otimes \mathrm{id}_3} & & & & \searrow^{\mathrm{id}_2 \otimes \tau_{1,3}} & \\
(V_1 \otimes V_2) \otimes V_3 & & & & & & V_2 \otimes (V_3 \otimes V_1) \\
 & \searrow_{\alpha_{1,2,3}} & & & & \nearrow_{\alpha_{2,3,1}} & \\
 & & V_1 \otimes (V_2 \otimes V_3) & \xrightarrow[\tau_{1,23}]{} & (V_2 \otimes V_3) \otimes V_1 & &
\end{array}$$

Braiding (cont.)

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Definition

The category of $U_q(\mathfrak{sl}_2)$-modules is said to admit a braiding

$$\{\tau_{U,V} : U \otimes V \to V \otimes U \mid U, V \text{ are } U_q(\mathfrak{sl}_2)\text{-modules}\}$$

if each $\tau_{U,V}$ is an invertible $U_q(\mathfrak{sl}_2)$-linear map, and if for any three $U_q(\mathfrak{sl}_2)$-modules U, V, W the above hexagon diagrams commute. We further call the braiding symmetric if

$$\tau_{V,U} \circ \tau_{U,V} = \mathrm{id}_{U \otimes V}$$

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$$\left(\sum_{n=0}^{\infty}(-1)^n q^{-n(n-1)/2}\frac{(q-q^{-1})^n}{[n]^!}F^n \otimes E^n\right)\circ \tilde{f}\circ \tau$$

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$\tilde{f}$ is essentially a scaling factor, which scales each eigenspace of the corresponding tensor factor by an appropriate value.

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- Taking the classical limit $q \to 1$.
- Modular tensor categories when q is a primitive root of unity.

...Now What?

- The construction of general quantum groups $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a Kac-Moody algebra, as well as a Lie superalgebra.
- Ribbon categories and ribbon Hopf algebras.
  - *Quantum Invariants of Knots and 3-Manifolds* by V.G. Turaev
- Hecke algebras and the Braid group.
- Statistical Mechanics and Quantum Yang-Baxter equation.
- Algebraic Combinatorics and nonsymmetric Macondald Polynomials
- Taking the classical limit $q \to 1$.
- Modular tensor categories when q is a primitive root of unity.
- Crystal bases and letting $q \to 0$.

Thank You!

Thank You! **applause**